

Affine Invariant Submanifolds with Completely Degenerate Kontsevich-Zorich Spectrum

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Abstract

We prove that if the Lyapunov spectrum of the Kontsevich-Zorich cocycle over an affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifold is completely degenerate, i.e. $\lambda_2 = \dots = \lambda_g = 0$, then the submanifold must be an arithmetic Teichmüller curve in the moduli space of Abelian differentials over surfaces of genus three, four, or five. As a corollary, we prove that there are at most finitely many such Teichmüller curves.

1 Introduction

In this paper we consider the Lyapunov exponents of the Kontsevich-Zorich cocycle on the absolute cohomology bundle over affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifolds of the moduli space of Abelian differentials on genus g surfaces. These exponents were studied extensively in [For02], [AV07], and [EKZ11]. In [For02], it was proven that the smallest non-negative exponent is always positive with respect to the canonical measures on strata of the moduli space of Abelian differentials. In [AV07], the Kontsevich-Zorich conjecture was verified using techniques independent of [For02], i.e. the spectrum of $2g$ exponents of the cocycle is simple with respect to the canonical measures on strata. Explicit formulas for sums of the positive Lyapunov exponents of the Kontsevich-Zorich cocycle were given in [EKZ11]. An affine invariant submanifold with $\lambda_2 = \dots = \lambda_g = 0$ is said to have completely degenerate KZ-spectrum. In this paper, we prove:

Theorem 5.5. *If \mathcal{M} is an affine invariant submanifold with completely degenerate KZ-spectrum, then \mathcal{M} is an arithmetic Teichmüller curve.*

In [For06], Forni gives an example of a genus three square-tiled surface generating a Teichmüller curve with completely degenerate KZ-spectrum. In [FM08], Forni and Matheus found an example of a genus four square-tiled surface generating a Teichmüller curve that also has completely degenerate KZ-spectrum. Möller [Möl11] found that these are the only examples of Teichmüller curves with completely degenerate KZ-spectrum with possible exceptions in the moduli space of Abelian differentials on genus five surfaces. In [Aul12], it was proven

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that with respect to any $\mathrm{SL}_2(\mathbb{R})$ orbit closure, the Teichmüller curves in genus three and four are the only such $\mathrm{SL}_2(\mathbb{R})$ orbits with completely degenerate KZ-spectrum in those genera. Furthermore, it was shown that for $g \geq 13$, there are no such orbits supporting an $\mathrm{SL}_2(\mathbb{R})$ -invariant measure. In [EKZ11], it was shown that there are no such regular affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifolds for $g \geq 7$, where regular is a technical condition defined in [EKZ11, § 1.5]. It was also shown in [Aul12] that if there are no Teichmüller curves with completely degenerate KZ-spectrum, as was conjectured in [Möl11], then there are no affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifolds with completely degenerate KZ-spectrum for $g \geq 5$.

Combining Theorem 5.5 with the results recalled above implies that the only affine invariant submanifolds with completely degenerate KZ-spectrum are the two known Teichmüller curves in genus three and four, and any other such affine invariant submanifold must be an arithmetic Teichmüller curve in genus five. Furthermore, the proof of Theorem 5.5 yields the following corollary.

Corollary 5.6. *There are at most finitely many Teichmüller curves with completely degenerate KZ-spectrum.*

The results in this paper rely on the recent fundamental work of Eskin and Mirzakhani [EM12], and Eskin, Mirzakhani, and Mohammadi [EMM12], which established that all $\mathrm{SL}_2(\mathbb{R})$ -orbit closures are affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifolds. Furthermore, the results of Avila, Eskin, and Möller [AEM12], Wright [Wri12], Möller [Möl11], and the author [Aul12] are essential ingredients in the proof of Theorem 5.5.

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2 Preliminaries

Strata of Abelian Differentials: Let X be a Riemann surface of genus $g \geq 2$ carrying an Abelian differential ω . If ω is holomorphic, then it has zeros of total order $2g - 2$. Let κ denote a partition of $2g - 2$. We consider the set of all pairs (X, ω) , where the orders of the zeros of ω are prescribed by κ . This set, denoted $\mathcal{H}(\kappa)$ is called a *stratum of Abelian differentials*. We assume throughout that (X, ω) has unit area with respect to the area form induced by ω .

$\mathrm{SL}_2(\mathbb{R})$ Action: An Abelian differential ω induces a flat structure on X . There is a natural action by $\mathrm{SL}_2(\mathbb{R})$ on this structure defined by embedding its charts in \mathbb{R}^2 and multiplying by an element of $\mathrm{SL}_2(\mathbb{R})$. Furthermore, the action preserves

the area of X with respect to ω . The action by the subgroup of diagonal matrices is called the *Teichmüller geodesic flow*.

Period Coordinates: Let Σ denote the set of zeros of ω . If we fix a basis $\{\gamma_1, \dots, \gamma_k\}$ for $H_1(X, \Sigma, \mathbb{Z})$, then we get a local map into relative cohomology

$$\Phi(X, \omega) := \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right) \in H^1(X, \Sigma, \mathbb{C}).$$

Orbit Closures: It was proven in [EM12] and [EMM12], that the closure of every $\mathrm{SL}_2(\mathbb{R})$ orbit in $\mathcal{H}(\kappa)$ is an affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifold \mathcal{M} that admits a finite $\mathrm{SL}_2(\mathbb{R})$ -invariant measure ν , which is affine with respect to period coordinates. For simplicity, we abbreviate the term *affine $\mathrm{SL}_2(\mathbb{R})$ -invariant submanifold* by *AIS*.

The tangent space of \mathcal{M} can be given in period coordinates as a subspace $T_{\mathbb{C}}(\mathcal{M})$. It satisfies $T_{\mathbb{C}}(\mathcal{M}) = \mathbb{C} \otimes T_{\mathbb{R}}(\mathcal{M})$, where $T_{\mathbb{R}}(\mathcal{M}) \subset H^1(X, \Sigma, \mathbb{R})$. Let $p : H^1(X, \Sigma, \mathbb{R}) \rightarrow H^1(X, \mathbb{R})$ be the natural projection to absolute cohomology.

If $\dim T_{\mathbb{R}}(\mathcal{M}) = 2$, then \mathcal{M} is called a *Teichmüller curve* and any surface (X, ω) generating a Teichmüller curve is called a *Veech surface*. (This is not the definition of a Veech surface, but a theorem of John Smillie.) If the Veech surface is square-tiled, then it generates an arithmetic Teichmüller curve.

Lyapunov Exponents: The bundle $H_{\mathbb{F}}^1$ over $\mathcal{H}(\kappa)$ is the bundle with fibers $H^1(X, \mathbb{F})$ and a flat connection given by identifying nearby lattices $H^1(X, \mathbb{Z})$ and $H^1(X', \mathbb{Z})$. If \mathcal{M} is an AIS, then the Teichmüller geodesic flow acts on every point in \mathcal{M} and thus, induces a flow on $H_{\mathbb{R}}^1$. This flow is known as the *Kontsevich-Zorich cocycle* (KZ-cocycle).

If we consider orbits under the Teichmüller geodesic flow that return infinitely many times to a neighborhood of the starting point, then it is possible to compute the monodromy matrix $A(t)$ at each return time t . By computing the logarithms of the eigenvalues of the $A(t)A^T(t)$, and letting t tend to infinity, we get a collection of $2g$ numbers known as the *spectrum of Lyapunov exponents of the KZ-cocycle*. By the Oseledec's multiplicative ergodic theorem, these numbers will not depend on the initial starting point for ν -almost every choice of initial data. Since cohomology admits a symplectic basis, this spectrum is symmetric and therefore it suffices to consider the set of g non-negative Lyapunov exponents of the KZ-cocycle

$$1 = \lambda'_1 \geq \dots \geq \lambda'_g \geq 0,$$

which will be known as the *KZ-spectrum*. We will suppress the measure from now on and always assume it to be the canonical measure guaranteed by [EM12].

If the Lyapunov exponents of the Kontsevich-Zorich cocycle over an AIS \mathcal{M} satisfy

$$1 = \lambda_1 > \lambda_2 = \dots = \lambda_g = 0,$$

then \mathcal{M} has *completely degenerate KZ-spectrum*. The first such example in genus three was found in [For06] and another example in genus four by [FM08]. Further results on this question nearing a complete classification of all such AIS with completely degenerate KZ-spectrum were established in [Mö11], [EKZ11], and [Aul12].

Forni Subspace: The *Forni subspace* $F(x) \subset H^1(X, \mathbb{R})$ was formally defined in [AEM12]. The subspace $F(x)$ is the maximal $\mathrm{SL}_2(\mathbb{R})$ -invariant subspace on which the KZ-cocycle acts by isometries with respect to the Hodge inner product. It was proven in [AEM12, Thm. 1.3] that for ν -a.a. x , $p(T_{\mathbb{R}}(\mathcal{M}))(x)$ is orthogonal to $F(x)$ with respect to the Hodge inner product.

In the language of [For06] and [Aul12], the corank of the rank k locus is equal to half the dimension of the Forni subspace, e.g. the rank one locus corresponds to the Forni subspace having maximal dimension $2g - 2$. One significant difference in the definition of these two concepts is that the rank k locus is a subset of the moduli space of Abelian differentials that is *not* $\mathrm{SL}_2(\mathbb{R})$ -invariant, whereas the Forni subspace is a subbundle of $H_{\mathbb{R}}^1$ that is $\mathrm{SL}_2(\mathbb{R})$ -invariant by definition. Furthermore, “rank” in the term “rank k locus,” refers to the rank of the derivative of the period matrix. The derivative of the period matrix is implicit in this paper via the Forni subspace, but will not be used explicitly, so we refer the reader to [Aul12, § 3] for an explanation of that perspective.

The key lemma for this paper which translates the language of [Aul12], into the language which will be used from here on, is Lemma 2.1. This lemma is stated in [Aul12, § 3.1]. However, it is a trivial consequence of [For06, Cor. 7.1]. By definition of the Forni subspace, having maximal dimension implies that the KZ-spectrum has $g - 1$ zero exponents. On the other hand, if the KZ-spectrum is completely degenerate, then the Forni-Kontsevich formula [For02, Cor. 5.3] forces the Forni subspace to have maximal dimension.

Lemma 2.1. *An AIS \mathcal{M} has completely degenerate KZ-spectrum if and only if its Forni subspace has maximal dimension, i.e. for a.e. $x \in \mathcal{M}$, $\dim_{\mathbb{R}}(F(x)) = 2g - 2$.*

Field of Definition: In [Wri12], the field of definition of an AIS was introduced. The *field of definition*, $\mathbf{k}(\mathcal{M})$ of an AIS \mathcal{M} is the smallest subfield of \mathbb{R} such that \mathcal{M} can be defined in local period coordinates by linear equations in $\mathbf{k}(\mathcal{M})$. It was proven that this is well-defined for every AIS and has degree at most g over \mathbb{Q} , [Wri12, Thm. 1.1].

Optimal Torus Coverings: In [Mö11], it was shown that every Teichmüller curve with completely degenerate KZ-spectrum is arithmetic, i.e. it is generated by a square-tiled surface. Square-tiled surfaces (X, ω) naturally cover the torus (\mathbb{T}^2, dz) . Let $\pi_{opt} : X \rightarrow \mathbb{T}^2$ denote the torus covering of minimal degree. It was shown in [Mö11] that for arithmetic Teichmüller curves with completely degenerate KZ-spectrum the degree of π_{opt} depends only on the stratum in

which the Teichmüller curve lies. Moreover, [Möl11] calculated this degree d_{opt} explicitly.

2.1 Cylinder Configuration

In this section we recall a result from [Aul12], which describes the cylinder decomposition of a surface whose AIS has completely degenerate KZ-spectrum. Then we choose a convenient homology basis on this surface that will be used in Lemma 3.1 below. The claims of this section are summarized in Figure 1.

Definition. *Given (X, ω) , let \mathcal{F}_θ denote the vertical foliation of $(X, e^{i\theta}\omega)$. If, for all $\theta \in \mathbb{R}$, the existence of a closed regular trajectory in \mathcal{F}_θ implies that every trajectory in \mathcal{F}_θ is closed, then (X, ω) is completely periodic.*

Remark. *This definition of “completely periodic” is consistent with that of [Aul12] and [Cal04], but different from that of [SW04].*

It is proven in [Aul12, Thm. 5.5], that any (X, ω) generating an AIS with completely degenerate KZ-spectrum, is completely periodic, though it was not phrased with this terminology because it was not available at the time. The cylinder configuration described in Corollary 2.2 is depicted in Figure 1. In the specific case of Veech surfaces, this corollary is [Möl11, Lem. 5.3].

Cylinder Index Notation: Throughout this paper the indices on the cylinders are taken modulo r so that we implicitly assume $C_{r+1} = C_1$.

Corollary 2.2 ([Aul12]). *Let (X, ω) generate an AIS with completely degenerate KZ-spectrum. For each $\theta \in \mathbb{R}$ such that the vertical foliation of $(X, e^{i\theta}\omega)$ is periodic, $(X, e^{i\theta}\omega)$ decomposes into a union of cylinders C_1, \dots, C_r such that each saddle connection on the top of C_i is identified to a saddle connection on the bottom of C_{i+1} and vice versa, for all $1 \leq i \leq r$. Furthermore, the circumference of C_i equals the circumference of C_j , for all i, j .*

The proof of Corollary 2.2 follows from considering any periodic direction, which exists because (X, ω) is completely periodic, and degenerating the surface by pinching the core curves of every cylinder in that direction. The configuration of the connected components of the resulting degenerate surface must be arranged in a cycle in order for the derivative of the period matrix to have rank one. This implies that the identification scheme in Corollary 2.2. Furthermore, no two saddle connections on the top of C_i (or the bottom of C_{i+1}) can be identified to each other without contradicting the orientability of a foliation transverse to the vertical foliation of ω .

Finally, we introduce a basis $\mathcal{B}(X, \omega) \subset H_1(X, \mathbb{Z})$, which is partially depicted in Figure 1, for the homology space of X . This basis is chosen in a way that is dependent on ω , and will be abbreviated \mathcal{B} when (X, ω) is understood. The cycle a_1 represents the family of core curves of the cylinders, which are pairwise homologous by Corollary 2.2. Also by Corollary 2.2, for each j , with $2 \leq j \leq g$,

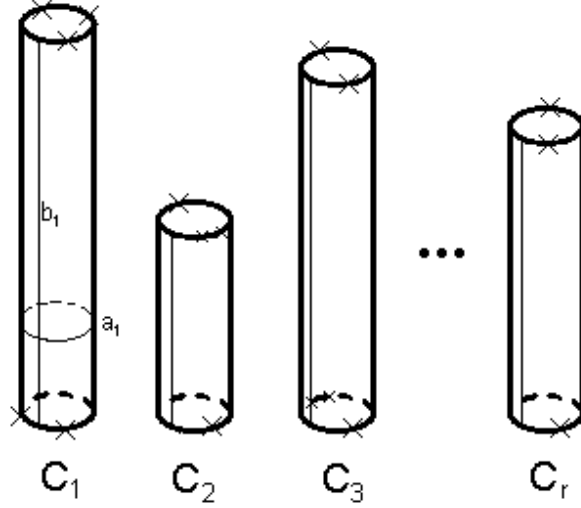


Figure 1: Homology Basis: Saddle connections on the top of C_i are identified with saddle connections on the bottom of C_{i+1} . Crosses denote copies of zeros.

it suffices to let each cycle a_j be a curve lying entirely in a small neighborhood of the boundaries of two adjacent cylinders defined so that $a_i \cap a_j = \emptyset$, for $i \neq j$. To see that such a choice is possible, cut each cylinder along its core curve thereby separating (X, ω) and collapse each core curve to a point to get a collection of positive genus surfaces (M_i, η_i) , $1 \leq i \leq r$. Each surface (M_i, η_i) admits a homology basis with a -cycles that are pairwise disjoint. Since the two discs (formerly half-cylinders) forming (M_i, η_i) are homologically trivial, the a -cycles must lie in a small neighborhood of the boundaries of the discs. Hence, they lie in a small neighborhood of the boundaries of the cylinders on (X, ω) .

The choice of b -cycles will not matter except for the cycle b_1 , which, by necessity, must traverse the height of every cylinder once in order to intersect the cycle a_1 exactly once. Define $\mathcal{B}(X, \omega) = \{a_j, b_j | 1 \leq j \leq g\}$.

3 Field of Definition

Lemma 3.1. *Let \mathcal{M} be an AIS with completely degenerate KZ-spectrum. Then $k(\mathcal{M}) = \mathbb{Q}$.*

Proof. Let $(X, \omega) \in \mathcal{M}$. Then (X, ω) is completely periodic by [Aul12, Thm. 5.5]. Without loss of generality, assume the horizontal foliation of (X, ω) is periodic. Furthermore, any periodic foliation can be written as a union of r cylinders of unit circumference, where r depends on the foliation, and the cylinders are arranged as in Corollary 2.2, see Figure 1. We choose a specific closed loop in \mathcal{M} that will simplify our calculations of the monodromy matrix below.

Consider the orbit of (X, ω) under the horocycle flow. By [SW04, Prop. 4(2)], the closure of this orbit is isomorphic to a d -dimensional torus, \mathbb{T}^d , for some d . If we consider an $\varepsilon > 0$ ball B about (X, ω) in $\mathbb{T}^d \subset \mathcal{M}$, then there is a sufficiently large value of $T \gg 1$ such that $h_T \cdot (X, \omega) := (X_1, \omega_1) \in B$ and the following property holds. Since we reach (X_1, ω_1) from (X, ω) by the horocycle flow, both (X, ω) and (X_1, ω_1) admit cylinder decompositions that differ only by the twisting of the cylinders themselves. Therefore, it is possible to consider the continuous path $\gamma' : [0, 1] \rightarrow B$, from (X_1, ω_1) to (X, ω) lying entirely in \mathbb{T}^d because \mathbb{T}^d is a closed manifold. We assumed T to be sufficiently large so that following the path γ' will not undo the twists induced by the horocycle flow. Let γ denote the path along the trajectory of the horocycle flow followed by concatenation with γ' .

Next we compute the monodromy matrix, denoted by A , of $[\gamma] \in \pi_1(\mathcal{M})$. To do this, we pass freely between cohomology and homology by Poincaré duality. Fix the homology basis \mathcal{B} defined in Section 2.1 and compute the monodromy matrix induced by following γ . For each of the r cylinders, traveling along γ results in C_k twisting $\rho_k \geq 0$ times, with $\rho_k \in \mathbb{Z}$. It is crucial to note that the assumption above on T being sufficiently large guarantees that there exists a value of k such that $\rho_k > 0$. Traveling along γ adds an a_1 cycle to every homology cycle traversing the height of every cylinder, while every other cycle is fixed.

By our choice of a -cycles, all of the a -cycles are fixed along the path γ , so that block of the monodromy matrix is the identity and the off-diagonal block that records the addition of b -cycles to a -cycles is the zero matrix. Furthermore, the twisting does not combine any pair of b -cycles and so that block of the monodromy matrix is also the identity. This follows because no linear combination of b -cycles is equal to a_1 . Therefore the monodromy matrix A corresponding to the path $[\gamma]$ is given as follows: let the first g rows and columns of A correspond to the a -cycles and the second set of g rows and columns of A correspond to the b -cycles. Then,

$$A = \begin{bmatrix} \text{Id} & 0 \\ M & \text{Id} \end{bmatrix},$$

where each block of A is a $g \times g$ matrix, and M will be determined below.

We claim that $M \neq 0$, which will imply that A is unipotent and $A \neq \text{Id}$. It suffices to find a single entry in M that is non-zero. Recall that the cycle $b_1 \in \mathcal{B}$ was chosen so that it traversed the height of every cylinder. Let $\rho = \sum_k \rho_k$ and recall that $\rho > 0$. Therefore, following the path γ will add ρ copies of a_1 to b_1 , which implies that the 1, 1 entry of M is $\rho \neq 0$.

Since A is unipotent, it generates a non-compact subgroup of matrices in $\text{SL}_{2g}(\mathbb{Z})$. If we consider the Jordan form of A , then there is at least one Jordan block of dimension at least 2×2 that is unipotent because matrix conjugation preserves unipotence. By [AEM12, Thm. 1.3] and Lemma 2.1, if ν is a finite affine measure on \mathcal{M} , then for ν -a.a. $x \in \mathcal{M}$, $H^1(X, \mathbb{R}) = p(T_{\mathbb{R}}(\mathcal{M}))(x) \oplus F(x)$. By the definition of $F(x)$, every monodromy matrix of \mathcal{M} must decompose into blocks such that there is a block of dimension $2g - 2 \times 2g - 2$ lying in $\text{SO}(2g -$

2). This implies that A is conjugate to a matrix whose Jordan decomposition consists of a 2×2 unipotent matrix and a $2g - 2 \times 2g - 2$ diagonal matrix with unit eigenvalues.

By [Wri12, Thm. 1.4], we have a decomposition of H^1 into subbundles $(\oplus_{\rho} \mathbb{V}_{\rho}) \oplus \mathbb{W}$, where the direct sum of \mathbb{V}_{ρ} is taken over all Galois conjugates ρ of $\mathbf{k}(\mathcal{M})$. This induces a decomposition of A into blocks corresponding to each of these subbundles. By [Wri12, Thm. 1.4], $\mathbb{V}_{\text{Id}} = p(T_{\mathbb{R}}(\mathcal{M}))$. Also, by [Wri12, Thm. 1.4] and the fact that $\text{SO}(n)$ does not contain any unipotent elements, the block corresponding to the subbundle \mathbb{V}_{Id} must be the 2×2 unipotent block. Furthermore, Galois conjugates of a unipotent block are unipotent. However, any non-trivial Galois conjugate of \mathbb{V}_{Id} would have to send the 2×2 unipotent block into the block contained in the compact group $\text{SO}(2g - 2)$. Such a contradiction implies that there can be no nontrivial Galois conjugates of $\mathbf{k}(\mathcal{M})$. Thus, $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. \square

Corollary 3.2. *If \mathcal{M} is an AIS with completely degenerate KZ-spectrum, then it contains a dense subset of arithmetic Teichmüller curves.*

Proof. By Lemma 3.1, $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. By the definition of $\mathbf{k}(\mathcal{M})$, \mathcal{M} can be written locally as the zero set of a set of linear equations with coefficients in \mathbb{Q} . Hence, it admits a dense set of rational solutions. Each rational solution corresponds to a square-tiled surface because the periods lie in $(1/q)\mathbb{Z}$, where $q \in \mathbb{Z}^*$ is the lowest common denominator of the rational solution. Finally, the arithmetic Teichmüller curves must be dense because the square-tiled surfaces are dense. \square

4 The Map π

We begin with a technical lemma. Let \mathcal{M} be an AIS. Given $(X, \omega) \in \mathcal{M}$, define the set

$$L(X, \omega) := \left\{ \int_{\gamma_i} \omega \mid \gamma_i \in H_1(X, \mathbb{Z}) \right\}.$$

Lemma 4.1. *Let \mathcal{M} be an AIS with $\dim p(T_{\mathbb{R}}(\mathcal{M})) = 2$ and $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. If $(X, \omega) \in \mathcal{M}$, then $L(X, \omega)$ is a discrete lattice in \mathbb{C} .*

Proof. Since \mathcal{M} is an AIS, locally identify \mathcal{M} with its tangent space $T_{\mathbb{R}}(\mathcal{M})$. Let $T_{\mathbb{R}}(\mathcal{M})$ have codimension d . Since $T_{\mathbb{R}}(\mathcal{M})$ is a linear subspace in period coordinates, there is a matrix $A \in \text{Mat}_{d \times k}(\mathbf{k}(\mathcal{M}))$ such that locally $\ker(A) = T_{\mathbb{R}}(\mathcal{M})$. Note that the natural map $p : H^1(X, \Sigma, \mathbb{R}) \rightarrow H^1(X, \mathbb{R})$ is a linear transformation of vector spaces. Hence, $p(\ker(A))$ is in fact a subspace of $H^1(X, \mathbb{R})$. Therefore, there exists another matrix A' such that $\ker(A') = p(T_{\mathbb{R}}(\mathcal{M}))$, and by assumption, $A' \in \text{Mat}_{2g-2 \times 2g}(\mathbb{Q})$. Without loss of generality, let $A' := (a_{ij})$ be in reduced row echelon form.

Fix a basis $\{\gamma_1, \dots, \gamma_k\}$ for $H_1(X, \Sigma, \mathbb{Z})$ so that $\{\gamma_1, \dots, \gamma_{2g}\}$ forms a basis for the absolute homology $H_1(X, \mathbb{Z})$. This induces a basis $\{\gamma_1^*, \dots, \gamma_k^*\}$ on cohomology by Poincaré duality so that $\{\gamma_1^*, \dots, \gamma_{2g}^*\}$ forms a basis for $H^1(X, \mathbb{Z})$.

If we consider period coordinates with respect to this basis, we get

$$\Phi(X, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right) \in \mathbb{C}^k.$$

Under the map p , we get

$$p \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_{2g}} \omega \right) := y.$$

Then y satisfies $A'y = 0$. Hence, for each $1 \leq i \leq 2g$,

$$\int_{\gamma_i} \omega = a_{i,1} \int_{\gamma_1} \omega + a_{i,2} \int_{\gamma_2} \omega,$$

where $a_{ij} \in \mathbb{Q}$. Let p be a positive integer such that $pa_{ij} \in \mathbb{Z}$, for all i, j . This implies that every element in pL is an integral linear combination of two complex numbers, thus pL is a lattice, as is L . \square

By Lemma 2.1, if \mathcal{M} is an AIS with completely degenerate KZ-spectrum, then $\dim p(T_{\mathbb{R}}(\mathcal{M})) = 2$, and Lemma 3.1 implies $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. This yields

Corollary 4.2. *Let \mathcal{M} be an AIS with completely degenerate KZ-spectrum. If $(X, \omega) \in \mathcal{M}$, then $L(X, \omega)$ is a discrete lattice in \mathbb{C} .*

Let $(X, \omega) \in \mathcal{M}$, where \mathcal{M} is an AIS with completely degenerate KZ-spectrum, and thus by Lemma 3.1, $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. Let $z_0 \in X$. Define the function

$$\pi(z) := \int_{z_0}^z \omega \mod L.$$

This function is well-defined because integrals along paths on X are defined up to integrals along absolute homology classes. Therefore, after modding out by this discrepancy, i.e. L , which in our case forms a lattice in \mathbb{C} by Corollary 4.2, we get a map from X to the torus \mathbb{T}^2 . We show that restricting to the subset of \mathcal{M} consisting of square-tiled surfaces, the map π agrees with the map π_{opt} up to the finite set of branch points, which are forgotten under π .

Lemma 4.3. *Let \mathcal{M} be an AIS with completely degenerate KZ-spectrum. If $(X, \omega) \in \mathcal{M}$ is a square-tiled surface, then $\pi = \pi_{opt}$ up to isogeny, i.e. π is a covering map of the torus with no non-trivial intermediate toral covers.*

Proof. Write the torus \mathbb{T}^2 to which X is mapped under π_{opt} as a fundamental domain of $\mathbb{C}/\mathbb{Z}[i]$, and show that π maps absolute homology cycles to elements of $\mathbb{Z}[i]$. Every element of $H_1(X, \mathbb{Z})$ is represented by a closed curve and continuous maps send closed curves to closed curves. Hence, every element of the homology space descends to a closed curve in \mathbb{T}^2 . Every closed curve in the torus is homotopic to a linear combination of the horizontal and vertical curves in the torus, which are exactly elements of $\mathbb{Z}[i]$ under π . \square

Lemma 4.4. *Let $\{\pi_{opt}^{(n)}\}_{n=1}^{\infty}$ be a sequence of optimal branched coverings of a torus, each having degree d_{opt} . For any subsequence of $\{\pi_{opt}^{(n)}\}_{n=1}^{\infty}$ converging to a torus covering π' , π' has the same degree as each of the maps $\pi_{opt}^{(n)}$.*

Proof. Let $\pi_{opt}^{(n)} : (X_n, \omega_n) \rightarrow (\mathbb{T}_n^2, dz)$. Fixing the area of the torus (\mathbb{T}_n^2, dz) to be one, for all n , implies the area of (X_n, ω_n) is d_{opt} . By assumption, $\{\pi_{opt}^{(n)}\}_{n=1}^{\infty}$ is a sequence of holomorphic maps converging to a holomorphic map. This implies that (X_n, ω_n) is a sequence of surfaces of fixed area d_{opt} converging to a surface also carrying a holomorphic differential, and therefore, the area must be preserved along this sequence. This implies that the limit of these surfaces (X', ω') must also have area d_{opt} and π'_{opt} must be a covering map of (X', ω') to the torus, whose area is also preserved. This forces the degree of π'_{opt} to be d_{opt} , the ratio of the areas. \square

5 Reduction to Teichmüller Curves

Recall that there is exactly one Teichmüller curve with completely degenerate KZ-spectrum in each of the two moduli spaces of Abelian differentials over surfaces of genus three and four by [Möl11]. Furthermore, these are the only AIS with completely degenerate KZ-spectrum in those genera by [Aul12]. By Corollary 3.2, an AIS with completely degenerate KZ-spectrum must contain a Teichmüller curve, and it is easy to see that that Teichmüller curve must also have completely degenerate KZ-spectrum. Since such Teichmüller curves do not exist for $g \geq 6$, there are no AIS with completely degenerate KZ-spectrum for $g \geq 6$. Hence, it suffices to assume that X has genus five throughout this section.

Assumption: The surface X has genus five.

5.1 Review of Results for Teichmüller Curves

The results in this subsection summarize the results of [Möl11] that will be used in this paper. The following lemma is proven in [Möl11], and we present an alternate proof here.

Lemma 5.1. *A Teichmüller curve with completely degenerate KZ-spectrum is arithmetic, i.e. it is generated by a square-tiled surface.*

Proof. By Lemma 3.1, any Teichmüller curve with completely degenerate KZ-spectrum has rational field of definition. By definition, Teichmüller curves defined over \mathbb{Q} are arithmetic and are generated by square-tiled surfaces by [GJ00]. \square

Let $E_n = \mathbb{C}/(n\mathbb{Z}[i])$ be a torus. The lattice points of E_n are called n -torsion points. Any Veech surface (X, ω) with completely degenerate KZ-spectrum is square-tiled by Lemma 5.1. Hence, it admits an optimal covering $\pi_{opt} : X \rightarrow E_n$,

which is branched over finitely many points $B \subset E_n$. Let $E_n^* = E_n \setminus B$. The degree of π_{opt} is denoted by d_{opt} . In [Möl11, Cor. 5.15], Möller derives an explicit formula for d_{opt} depending only on the stratum in which (X, ω) lies. Möller proves that the only possible strata that can contain an arithmetic Teichmüller curve lie in genus three, four, and five. He gives a table with all possible values of d_{opt} .

A cylinder C in E_n^* is bounded by points in B and lifts to one or more cylinders in $\pi_{opt}^{-1}(C) \subset X$. The following lemma is [Möl11, Lem. 5.17] and it fully describes how cylinders lift to the Veech surface (X, ω) , where X has genus five.

Lemma 5.2 (Möller). *In each of the possible strata in genus five containing Teichmüller curves with completely degenerate KZ-spectrum, one of the following three possibilities holds.*

- (i) *The preimage under π_{opt} of each cylinder in E_n^* consists of only one cylinder in X , or*
- (ii) *B consists of one element only and each cylinder in E_n^* has k preimages under π_{opt} where $2 \leq k \leq 4$, or*
- (iii) *B is contained in the set of 2-torsion points of E_n^* . Moreover, each cylinder in E_n^* has two preimages under π_{opt} .*

5.2 Main Results

Lemma 5.3. *Any infinite family of Teichmüller curves $\{C'_n\}_{n=1}^\infty$ with completely degenerate KZ-spectrum contains an infinite family of curves $\{C_n\}_{n=1}^\infty$ such that for each n , there is a Veech surface $(X_n, \omega_n) \in C_n$ such that (X_n, ω_n) satisfies Case i) of Lemma 5.2.*

Proof. We show that there are at most finitely many examples of Teichmüller curves in genus five, not obeying the assumptions of this lemma. Consider Cases ii) and iii) of Lemma 5.2. In each of these cases, the branching points on the torus lie at either the origin or the 2-torsion points. Hence, the torus can be realized as a single square or four unit squares, respectively. This implies that the surface covering the torus can be realized by at most $4d_{opt}$ squares. Hence, there can be only finitely many such examples in these cases. \square

Lemma 5.4. *Given an infinite sequence of Teichmüller curves $\{C_n\}_{n=1}^\infty$ with completely degenerate KZ-spectrum in a fixed stratum, there exists a subsequence converging to a Teichmüller curve \mathcal{C} in an adjacent stratum whose KZ-spectrum is also completely degenerate.*

Proof. By [Möl11], $\{C_n\}_{n=1}^\infty$ is in a stratum of Abelian differentials on genus five surfaces. Using Lemma 5.3, we pass to an infinite family satisfying Case i). Every cylinder in \mathbb{T}^2 lifts to a unique cylinder in X by the defining property of Case i). Let $\{(X_n, \omega_n)\}_{n=1}^\infty$ be an infinite sequence of Veech surfaces, satisfying

$(X_n, \omega_n) \in \mathcal{C}_n$ and $\pi_{opt}(X_n, \omega_n) = (\mathbb{T}_n^2, dz)$, for all n . Consider the induced sequence $\{(\mathbb{T}_n^2, dz)\}_{n=1}^\infty$. Within this sequence, we can pass to a subsequence such that the minimum distance between two branch points on \mathbb{T}_n^2 is $h_n < 1/n$. Such a sequence can always be constructed by modifying the given sequence via an action on (X_n, ω_n) and (\mathbb{T}_n^2, dz) by $H^u(t)$ or $H^l(t)$, where

$$H^u(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, H^l(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

To fix a natural set of coordinates on the torus, consider the torus $\mathbb{T}_n^2 = \mathbb{C}/(m_n\mathbb{Z}[i])$, for a positive integer m_n , and assume that three of the branch points of \mathbb{T}_n^2 lie at the origin, on the real axis, and on the imaginary axis of \mathbb{C} . The horizontal and vertical directions on the torus induce cylinder decompositions of the torus with cylinders bounded by branch points, and cylinder decompositions of the surface X_n with cylinders bounded by zeros of ω . Since h_n was assumed to be a bound on the minimal distance between branch points and there are zeros lying above the branch points on X_n , the horizontal or vertical cylinder decomposition of X_n must have a cylinder C_n of height at most $1/n$. Both horizontal and vertical directions had to be considered in order to avoid the case in which the two branch points of distance h_n lie on the same horizontal or vertical line.

Let $H^*(t_n)$ denote either $H^u(t)$ or $H^l(t)$ depending on which one is needed to make the following argument work. In other words, if b and b' represent the two branch points of distance at most h_n , where b lifts to a zero on the top of C_n and b' lifts to a zero on the bottom of C_n , and b and b' lie on the same horizontal line in \mathbb{T}_n^2 , choose $H^*(t) = H^l(t)$. Otherwise, let $H^*(t) = H^u(t)$. Act on (X_n, ω_n) by $H^*(t_n)$ so that two of the zeros lying above b and b' on C_n have distance less than $1/n$ after twisting. Consider the sequence $\{H^*(t_n) \cdot (X_n, \omega_n)\}_{n=1}^\infty$ of Veech surfaces. The circumferences of the cylinders in the horizontal (resp. vertical) direction of each surface are fixed for all n . Therefore, as n tends to infinity, $H^*(t_n) \cdot (X_n, \omega_n)$ cannot converge to a surface carrying an Abelian differential with simple poles.

A priori, the sequence $\{H^*(t_n) \cdot (X_n, \omega_n)\}_{n=1}^\infty$ can degenerate to a surface carrying a holomorphic differential with more than one connected component. We claim that this is not possible. By Lemma 2.2, the only way for collapsing saddle connections to separate the surface is for two sets of saddle connections between two pairs of cylinders to simultaneously collapse to a point. If just some of the saddle connections along the top of a cylinder collapse to a point, the cylinder will still have its remaining saddle connections identified to the bottom of the next cylinder. However, it is impossible for every saddle connection to collapse to a point because the core curve of the flat cylinder is homotopic to the union of all of the saddle connections along the top of a cylinder, and pinching such a curve would result in a simple pole, which was already excluded by the argument above.

Therefore, if (X', ω') denotes the limit of the sequence $\{H^*(t_n) \cdot (X_n, \omega_n)\}_{n=1}^\infty$, then (X', ω') must be a connected surface carrying a holomorphic differential.

In [Aul12, Lem. 7.1], it is shown that such a surface (X', ω') must also have completely degenerate KZ-spectrum because the set of all surfaces in the moduli space with completely degenerate KZ-spectrum is closed, i.e. the rank one locus is closed. Finally, it is possible that a union of saddle connections forming a closed curve is pinched to a node on (X', ω') . However, this possibility is excluded by the main technical result of [Aul12, Lem. 9.15], which says that X' must have genus five. Hence, (X', ω') generates a Teichmüller curve in an adjacent stratum. \square

Theorem 5.5. *If \mathcal{M} is an AIS with completely degenerate KZ-spectrum, then \mathcal{M} is an arithmetic Teichmüller curve.*

Proof. By contradiction, assume that \mathcal{M} is not a Teichmüller curve. By Corollary 3.2, \mathcal{M} contains a dense set of arithmetic Teichmüller curves. By Lemma 5.4, there exists a sequence of Teichmüller curves in \mathcal{M} , converging to a Teichmüller curve \mathcal{C} in an adjacent stratum. By [Möl11, Cor. 5.15], the degree of each covering map π_{opt} from a square-tiled surface in \mathcal{M} to the torus is a fixed number d_{opt} . By Lemma 4.4, the covering map of the torus by a surface generating \mathcal{C} must also have degree d_{opt} . Hence, \mathcal{M} is an AIS in a stratum whose degree d_{opt} is equal to the degree of the covering in an adjacent stratum. However, by inspection of the table in [Möl11, Cor. 5.15], no two strata have the same d_{opt} . This contradiction implies that \mathcal{M} is a Teichmüller curve, and it is arithmetic by Lemma 5.1. \square

Note that the only property of \mathcal{M} on which the proof of Theorem 5.5 relied, was the existence of infinitely many Teichmüller curves with completely degenerate KZ-spectrum contained in \mathcal{M} . The contradiction of this property at the end of the proof of Theorem 5.5 also proves the following corollary.

Corollary 5.6. *There are at most finitely many Teichmüller curves with completely degenerate KZ-spectrum.*

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